Monin-Lundgren hierarchy versus the Hopf equation in the statistical theory of turbulence

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A functional-differential equation equivalent to the infinitely hierarchic system of the coupled equations for many-point velocity distribution functions in turbulence established by Monin and Lundgren is presented, to be compared with the Hopf characteristic functional equation which governs the time evolution of a turbulent velocity field. It turns out that this functional entirely implies the Hopf functional, and vice versa. This functional allows us to view turbulence in a Lagrangian picture simulated by many-particle dynamics.

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To treat turbulence as an ensemble of incompressible flow velocity fields governed by the Navier-Stokes equation, Hopf established a statistical hydromechanics [1] in which the probability measure of the velocity fields is a functional subject to a functional-differential equation. On the other hand, Monin [2] and Lundgren [3] developed independently the hierarchy of equations for multipoint velocity distribution functions in a turbulent flow field to grasp the above-described ensemble from another angle, seemingly in an easier way. Although turbulence has as yet been too difficult to approach by either method, it is important to recognize the exact mathematical relation between the two methods. People may expect that the so-called Monin-Lundgren (ML) hierarchy would reach the Hopf equation in the limit when the number of the points taken in the field goes to infinity. But how, and in what sense? This fundamental issue in fluid mechanics is still open. To investigate it, we need to unite the full infinite ML hierarchy into a single functional-differential equation. We shall use here the technique by which the infinite Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy for a many-particle system [4] was replaced by a single functional-differential equation [5].

In the ML hierarchy for a wall-free turbulence, the s-point velocity distribution function $F_s(x_1,\ldots,,x_s,t)$, where $x=(\mathbf{v},\mathbf{r})$, $\mathbf{v}$ is the velocity vector at the position $\mathbf{r}$, and t the time varible, is governed (for $s=1,2,\ldots$) by the equation

$$\frac{\partial F_s}{\partial t} = -\sum_{i=1}^{s}\frac{\partial}{\partial \mathbf{v}_i}\cdot\frac{\partial F_s}{\partial \mathbf{r}_i} + \sum_{i=1}^{s}\frac{\partial}{\partial \mathbf{v}_i} \cdot\left\{\frac{1}{4\pi}\int \frac{\partial}{\partial \mathbf{r}_i}\frac{1}{|\mathbf{r}_i-\mathbf{r}_{s+1}|}\left(\mathbf{v}_{s+1}\cdot\frac{\partial}{\partial \mathbf{r}_{s+1}}\right)^2 \times F_{s+1}d\mathbf{v}_{s+1}d\mathbf{r}_{s+1} - \lim_{\mathbf{r}_{s+1}\to\mathbf{r}_i} \times \nu\left(\frac{\partial}{\partial \mathbf{r}_{s+1}}\cdot\frac{\partial}{\partial \mathbf{r}_{s+1}}\right)\int \mathbf{v}_{s+1}F_{s+1}d\mathbf{v}_{s+1}\right\}, \tag{1}$$

where ν is kinematic viscosity of fluid. Obviously F_s is symmetric for an exchange of each argument x_i. Here, we introduce the generating functional,

$$\phi[y(x),t] = \sum_{s=0}^{\infty}\int\cdots\int \frac{i^s}{s!}F_s(x_1,\ldots,x_s,t) \times y(x_1)\ldots y(x_s)dx_1\ldots dx_s. \tag{2}$$

Here, $i=\sqrt{-1}$; this imaginary factor has been inserted for later convenience, and $y(x)$ is an arbitrary real function. We define $F_0=1$, which guarantees

$$\phi[0,t]=1. \tag{3}$$

Then, it is not hard to find from (1) that the time evolution of ϕ should obey the following single linear functional-differential equation:

$$\frac{\partial\phi}{\partial t} = \int iy(x)\left[-\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}\frac{\delta}{i\delta y(x)}\right]\phi dx + \int\int iy(x)\frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{\Omega}(x,x')\frac{\delta^2}{i^2\delta y(x)\delta y(x')}\phi dx dx' \tag{4}$$

with

$$\mathbf{\Omega}(x,x') = \frac{1}{4\pi}\frac{\partial}{\partial \mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(\mathbf{v}'\cdot\frac{\partial}{\partial \mathbf{r}'}\right)^2 - \nu\delta(\mathbf{r}-\mathbf{r}')\left(\frac{\partial}{\partial \mathbf{r}'}\cdot\frac{\partial}{\partial \mathbf{r}'}\right)\mathbf{v}'. \tag{5}$$

In other words, Eq. (1) is obtained for any s as the relation of the coefficient functions of the sth-order terms with respect to y in both sides of Eq. (4). Therefore, the infinite ML hierarchy is mathematically equivalent to Eq. (4) with (5).

In order for F_s to be the s-point velocity distribution, however, there are some conditions to be obeyed, as follows.

Reduction condition:

$$\int F_{s+1}(x_1,\ldots,x_{s+1},t)d\mathbf{v}_{s+1} = F_s(x_1,\ldots,x_s,t); \tag{6}$$

when $s=0$, the right-hand side is unity by definition. This gives the normalization condition of probability.

Coincidence condition:

$$\int F_{s+1}(x_1, \ldots, x_{s+1}, t)\,\delta(\mathbf{r}_s - \mathbf{r}_{s+1})\,d\mathbf{r}_{s+1} = F_s(x_1, \ldots, x_s, t)\,\delta(\mathbf{v}_s - \mathbf{v}_{s+1}); \tag{7}$$

this may be rewritten as

$$\int F_{s+1}(x_1, \ldots, x_{s+1}, t)\,\delta(\mathbf{r}_s - \mathbf{r}_{s+1})\,dx_{s+1} = F_s(x_1, \ldots, x_s, t). \tag{8}$$

Separation condition: When some points are very far apart from the others, the distribution function for them becomes independent of that for the others. If all points are apart this way, we must have

$$F_s(x_1, \ldots, x_s, t) = F_1(x_1, t) \ldots F_1(x_s, t). \tag{9}$$

Divergence condition: Since we treat an incompressible flow, the average velocity is divergence-free at any point,

$$\frac{\partial}{\partial \mathbf{r}_i} \cdot \int \mathbf{v}_i F_s(x_1, \ldots, x_s, t)\,d\mathbf{v}_i = 0, \tag{10}$$

for $1 \leqslant i \leqslant s$.

These conditions are expressed in terms of ϕ successively as

$$\int \frac{\delta\phi}{i\,\delta y(x)}\,d\mathbf{v} = \phi, \tag{6'}$$

$$\int \frac{\delta^2\phi}{i^2\,\delta y(x)\,\delta y(x')}\,\delta(\mathbf{r} - \mathbf{r}')\,d\mathbf{x}' = \frac{\delta\phi}{i\,\delta y(x)}, \tag{8'}$$

$$\phi = \exp[i/F_1(x,t)y(x)dx], \tag{9'}$$

$$\frac{\partial}{\partial \mathbf{r}} \cdot \int \mathbf{v}\frac{\delta\phi}{i\,\delta y(x)}\,d\mathbf{v} = 0. \tag{10'}$$

It is to be noted that they are all linear equations with respect to ϕ.

Now let us consider the functional Fourier transform of ϕ, using functional integration (see [5]) that may be defined as

$$p[f(x),t] = \int \exp\left[-i\int f(x)y(x)dx\right]\phi[y(x),t]\,\delta y, \tag{11}$$

where $f(x)$ is an arbitrary real function in x. Inversely, we may have

$$\phi[y(x),t] = \int \exp\left[i\int f(x)y(x)dx\right]p[f(x),t]\,\delta f. \tag{12}$$

If we insert this into the basic equation (3), we have *the first-order functional-differential equation*,

$$\frac{\partial p}{\partial t} = -\int \frac{\delta}{\delta f(x)}\left[-\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}f(x)\right]p\,dx - \int\int \frac{\delta}{\delta f(x)}\frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{\Omega}(x,x')f(x)f(x')p\,dx\,dx', \tag{13}$$

where we note that a partial functional integration (with respect to f)[5] has been used, assuming that $p \to 0$ as $|f| \to \infty$. This equation is fortunately much simpler than the counterpart for the BBGKY hierarchy because that is a second-order functional-differential equation [6]; the second-order functional-differential term which stems from the short-range interaction between particles in the Liouville dynamics and is essential to viewing the irreversible aspect of development of the system leading to the H theorem. There is not such a sophisticated situation here in (13).

It is apparent that Eq. (13) implies the *nondiffusive* conservation law of $p\,\delta f$ (as a measure) in the function space of f; the right-hand side indicates minus the functional-divergence of the flux of the measure density p. Therefore, the total prefactor function of this flux should give the time rate of an individual position f moving in the function space, that is

$$\frac{\partial f(x)}{\partial t} = -\mathbf{v}\cdot\frac{\partial f(x)}{\partial \mathbf{r}} + \int \frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{\Omega}(x,x')f(x)f(x')dx' \equiv Qf(x). \tag{14}$$

Note that Q is a nonlinear operator to $f(x)$. Conditions $(6)'-(10)'$ restrict $f(x)$ through Eq. (12) successively as follows:

$$\int f(x)d\mathbf{v} = 1, \tag{6''}$$

$$\int f(x)f(x')\,\delta(\mathbf{r} - \mathbf{r}')dx' = f(x), \tag{8''}$$

which is equivalent to (6″),

$$p[f(x),t] = \delta[f(x) - F_1(x,t)], \tag{9''}$$

which is a very rare case of p, and

$$\partial/\partial\mathbf{r}\cdot\int \mathbf{v}f(x)d\mathbf{v} = 0. \tag{10''}$$

Thus, Eq. (14) essentially associated with (6″) and (10″) is seemingly a kinetic equation for an ensemble of particles in the x space subject to a self-interactive force characterized by the operator $\mathbf{\Omega}(x,x')$; just like the Vlasov equation in kinetic theory [7]. Lundgren [3] called this "an effective field collisionless Boltzmann equation" as the first approximation to the equation for $F_1(x,t)$ [by neglecting the additional effect of $F_2(x,t)$] and showed that $\int\mathbf{\Omega}(x,x')f(x')dx'$ involves the pressure force and viscous resistance to a flow particle. It is easy to make sure in the dynamics (14) that conditions (6″) and (10″) together conserve with time if $f(x)$ at an initial time subordinates them, only bearing in mind that $f(x) \to 0$ as $|\mathbf{v}| \to \infty$, and that $(1/4\pi)\partial/\partial\mathbf{r}\cdot\partial/\partial\mathbf{r}(1/|\mathbf{r}-\mathbf{r}'|) = -\delta(\mathbf{r}-\mathbf{r}')$. And, these conditions are stronger than the originals $(1)'-(4)'$; hence $(1)'-(4)'$ should conserve during the time development of ϕ. On the other hand, it is clear from (3), (12), and (13) that $p[f(x),t]$ is a density functional of conservative probability measure on the function space of f, and then $\phi[y(x),t]$ is nothing but the corresponding characteristic

functional for the stochastic scalar field $f(x)$. We should note from (2) and (12) that

$$F_s(x_1, \ldots, x_s, t) = \int f(x_1) \ldots f(x_s) p[f(x), t] \delta f, \qquad (15)$$

meaning that the ensemble average of sth-order correlations of the scalar field should give exactly the s-point velocity distribution functions.

Here, we must see the distinction between us and Lundgren in methodology; $f(x)$ governed by Eq. (14) is not to be an approximation to F_1 but a member of the ensemble evolving according to Eq. (13), which can embrace all F_s by means of Eq. (15). Lundgren [8] approximately pursued $F_1(x,t)$ by bringing forth a modeled collision term into Eq. (14) in analogy with the Bhatnager-Gross-Krook model of gas kinetics, while there is no such collision mechanism found here, as is clear from the logical process leading to Eq. (14). Since the BBGKY hierarchy is fundamentally different from the ML hierarchy in mechanical structure, as was described above, it is questionable to employ in the latter system such a statistical-relaxation model (as a surrogate of the collision term in the Boltzmann equation, which helped the former system going towards local thermal equilibrium). Actually, the only short-range (nearest-neighbor) interaction of particles in the dynamics (14) of $f(x)$ must come through the viscous force, as will be discussed later, and this term only provides an irreversibility in turbulence in terms of dissipation.

Then, let us simply rewrite Eq. (4) as

$$\frac{\partial \phi[y(x),t]}{\partial t} = \int iy(x) Q\left[\frac{\delta}{i\,\delta y(x)}\right] \phi[y(x),t] dx. \qquad (16)$$

This is of the same form as the Hopf equation for turbulence except the operator Q is not the Navier-Stokes operator to the velocity field, and $y(x)$ is not a three-dimensional vector field but a six-dimensional scalar function. However, once $\phi[y(x),t]$ is solved, it can yield the equivalent to the Hopf characteristic functional, by specially limiting the argument into

$$y(x) = \mathbf{v} \cdot \boldsymbol{\theta}(\mathbf{r}), \qquad (17)$$

as

$$\boldsymbol{\Phi}[\boldsymbol{\theta}(\mathbf{r}),t] \equiv \phi[\mathbf{v} \cdot \boldsymbol{\theta}(\mathbf{r}),t] = \int \exp\left[i \int f(x) \mathbf{v} \cdot \boldsymbol{\theta}(\mathbf{r}) dx \right] p[f(x),t] \delta f. \qquad (18)$$

Since $\int f(x) \mathbf{v} d\mathbf{v} \equiv \mathbf{u}(\mathbf{r})$ can be considered as a divergence-free [due to (10″)] stochastic velocity vector field, judging from Eq. (15) and the definition of F_s, then $\boldsymbol{\Phi}[\boldsymbol{\theta}(\mathbf{r}),t]$ in (18) must be the characteristic functional for a velocity field $\mathbf{u}(\mathbf{r})$. Thus, $\phi[y(x),t]$ implies the Hopf functional in itself.

On the other hand, we can easily show that the Hopf functional implies $\phi[y(x),t]$, as well. Let us assume that we know $\boldsymbol{\Phi}[\boldsymbol{\theta}(\mathbf{r}),t]$, and set $\boldsymbol{\theta}(\mathbf{r})$ as

$$\boldsymbol{\theta}(\mathbf{r}) = \sum_{i=1}^{s} \boldsymbol{\theta}_i \delta(\mathbf{r} - \mathbf{r}_i). \qquad (19)$$

Then, Φ reduces to the s-point characteristic function of velocity at $\mathbf{r}_1, \ldots, \mathbf{r}_s$ like

$$\boldsymbol{\Phi}_s[\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_s, t] = \int \exp\left[i \sum_{i=1}^{s} \boldsymbol{\theta}_i \cdot \mathbf{u}(\mathbf{r}_i) \right] P[\mathbf{u},t] \delta \mathbf{u}, \qquad (20)$$

where $P[\mathbf{u},t]\delta\mathbf{u}$ is the differential probability measure of the space of velocity field $\mathbf{u}(\mathbf{r})$. It is clear that F_s in (2) is the inverse Fourier transform of $\boldsymbol{\Phi}_s$, and hence we can produce a full form of $\phi[y(x),t]$ in the limit of $s \to \infty$, using this knowledge from $\boldsymbol{\Phi}[\boldsymbol{\theta}(\mathbf{r}),t]$. Then, the equivalence of both functionals has been established here.

Based on the theorem of probability conservation in the function space as in the Hopf theory [1], the formal solution of Eq. (16) is given as

$$\phi[y(x),t] = \int \int \exp\left\{ i \int [y(x) T^{t-t_0} - y'(x)] f(x) dx \right\} \times \phi[y'(x), t_0] \delta y' \, \delta f, \qquad (21)$$

where T^{t-t_0} is the solution operator of Eq. (14) starting from $f(x)$ at the initial time t_0, such that

$$T^{t-t_0} f(x) = \lim_{N \to \infty} (1 + \Delta t Q)^N f(x), \qquad (22)$$

where $N\Delta t = t - t_0$. Indeed, it is evident that (21) satisfies the basic equation (16); that is because

$$\frac{\partial \phi[y(x),t]}{\partial t} = \int iy(x) \frac{\partial}{\partial t} [T^{t-t_0} f(x)] \{\phi \text{ in (21)}\} dx,$$

$$= \int iy(x) Q[T^{t-t_0} f(x)] \{\phi \text{ in (21)}\} dx, \qquad (23)$$

by means of Eq. (14), and the last equation is equal to (16).

Now, let us see what the content of our functional is. The function $f(x)$ governed by Eq. (14) is suitable to be called "density" of particles in the x space. In fact, the characteristic curves of Eq. (14),

$$d\mathbf{r}/dt = \mathbf{v},$$

$$d\mathbf{v}/dt = \int \boldsymbol{\Omega}(x,x') f(x') dx', \qquad (24)$$

are considered to give the Lagrangian paths of all the fluid particles moving in the x space. This may offer an approach to turbulence based on the concept of molecular dynamics, only if $f(x')$ in the right-hand side can be simulated by the smoothed density of extremely many particles in the x space at every time. But, the elementary dynamics between particles is far different from a usual one with the Boltzmann-type collision. Our interaction between particles is executed by nonconservative, nonlocal (pressure), and local (viscosity) forces.

Apart from practical utility, a crude idea of motion of N

particles in the system may be given by roughly expressing $f(x)$ in (14) as $\Sigma_i \delta[\mathbf{v}-\mathbf{v}_i(t)]U(\mathbf{r}-\mathbf{r}_i(t))/N$, where $U(\mathbf{r})=1$ for $r=0$ and otherwise vanishes. Then, it yields a closed dynamical system governed by

$$d\mathbf{r}_i/dt = \mathbf{v}_i,$$

$$d\mathbf{v}_i/dt = -\frac{1}{4\pi}\frac{\partial}{\partial \mathbf{r}_i}\sum_j{}' \frac{\partial}{\partial \mathbf{r}_j}\frac{\partial}{\partial \mathbf{r}_j}\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|}:\mathbf{v}_j\mathbf{v}_j/N + \nu\sum_j{}'' \frac{\mathbf{v}_j-\mathbf{v}_i}{|\mathbf{r}_j-\mathbf{r}_i|^2}, \tag{25}$$

where the first sum Σ' should avoid the self-interaction for $j=i$, and the second sum Σ'' is taken over only the six particles nearest (in the $\mathbf{r}$ space) to the ith particle; this sum approximates the Laplacian on the fictitious field $\mathbf{v}(\mathbf{r})$ near $\mathbf{r}=\mathbf{r}_i$, which is, indeed, a central-difference scheme for the Laplacian when the six points are located at the face-center positions of the cube involving the ith particle at its center. The initial condition for particles should be given as a discrete approximation to an arbitrary solenoidal smooth field $\mathbf{v}(\mathbf{r})$. The number N need be large enough for the average interparticle distance to be less than the Kolmogorov length. In such an approximation, it is convenient to adopt the cyclic boundary condition for the case of wall-free turbulence. This idea is, however, an extreme simplification and may not always be a good approach to Eqs. (24). But, it would be an easy-to-understand picture of the theoretical implication of our functional equivalent to the infinite ML hierarchy. Nevertheless this scheme of particle dynamics may tempt us to practice a direct calculation to investigate the problems of relative diffusion and mixing in turbulence, if with a capacious supercomputer.

In conclusion, we have presented a functional-differential equation equivalent to the infinite ML hierarchy and proved that the dependent functional in itself is the characteristic functional of a stochastic six-dimensional scalar function $f(x)$, and that it is equivalent to the Hopf characteristic functional of velocity field in turbulence.

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